The arbitrage-free equilibrium pricing of liabilities in an incomplete market: application to a South African retirement fund

Rob Thomson

Agenda

1. Introduction
2. Market-portfolio model
3. Equilibrium asset categories model
4. Liabilities specification & model
5. Pricing method
6. Results and sensitivity
7. Conclusions

Introduction

Aim:

Apply the pricing method of Thomson (2005) to:

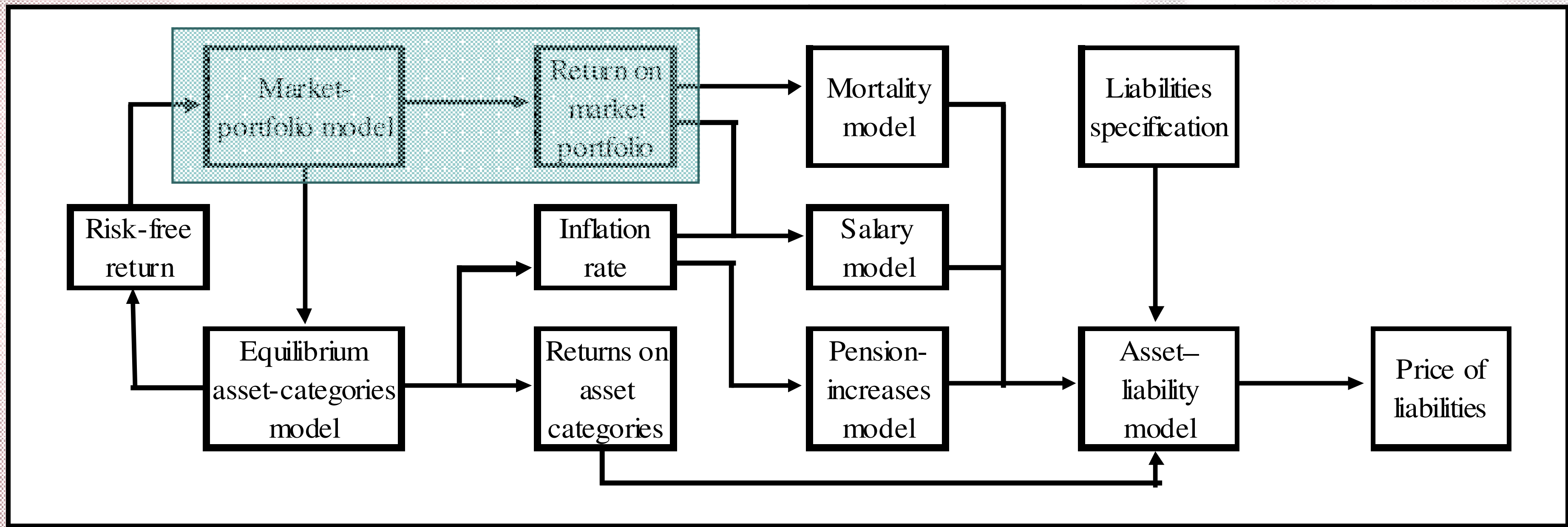
- the market-portfolio model of Thomson (unpublished);
- the equilibrium asset-category model of Thomson & Gott (forthcoming);
- a DB retirement-fund model;

with a view to operationalising the pricing of such a fund and quantifying the effects of:

- the risk premium due to incompleteness;
- non-additivity due to incompleteness;
- guarantees implicit in reasonable expectations of pension increases; and
- the sensitivity of the price of illustrative liabilities to the parameters of the model.

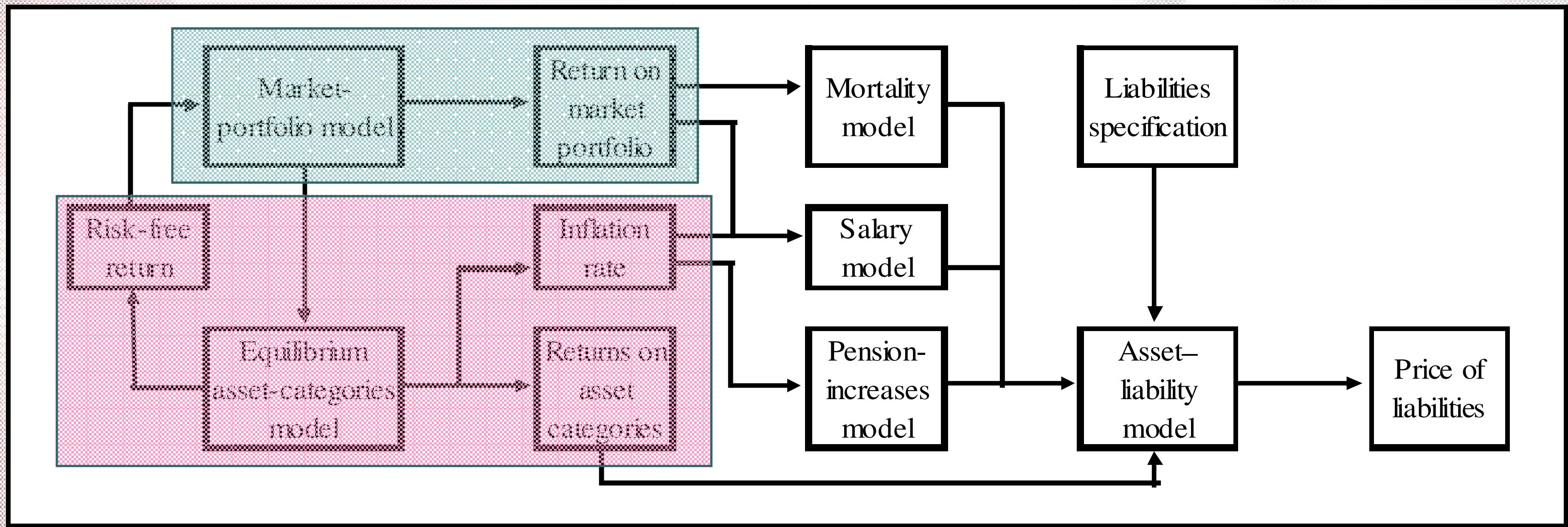
Introduction

Method:



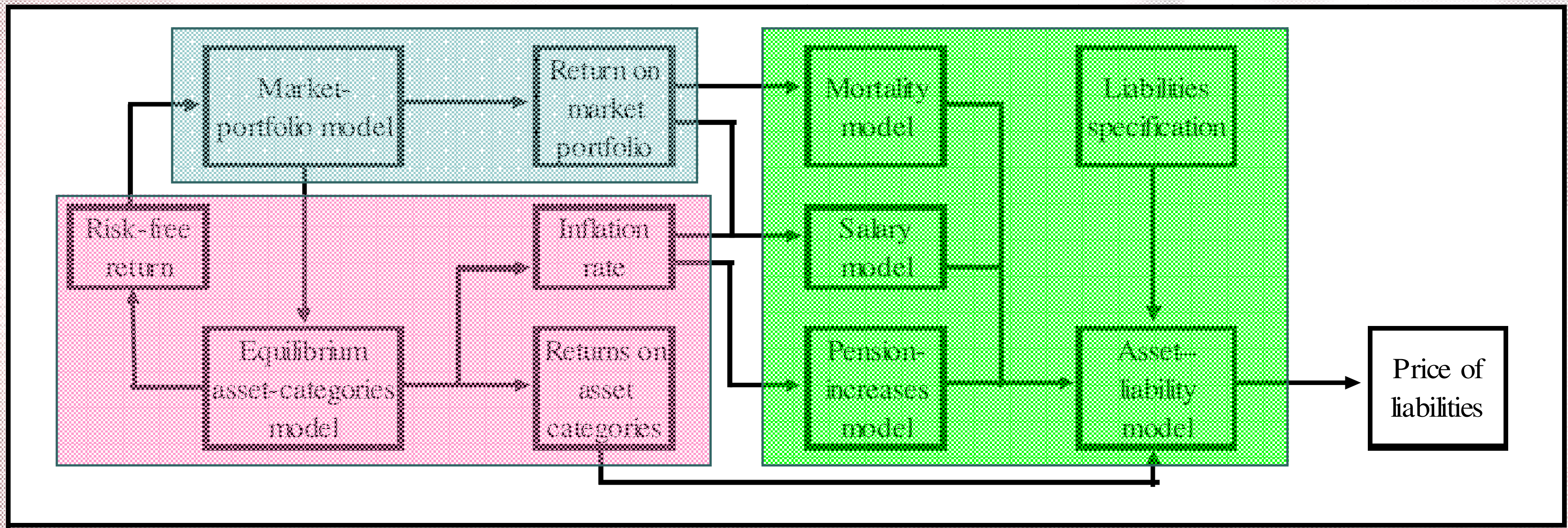
Introduction

Method:



Introduction

Method:



Introduction

- Time is measured in intervals of one year.
- The ‘return’ during a year is the average instantaneous real rate (or force) of return during that year.

Market-portfolio model

$$\delta_{Mt} = \mu_{Mt} + \sigma_M \varepsilon_t \qquad \sigma_M = 0{,}159$$

where:

$$\mu_{Mt} = g\,\delta_{It}(0) \text{ for } \delta_{It}(0) > 0; \qquad g = 1{,}39$$

$$\delta_{It}(0) \text{ otherwise}$$

$$\varepsilon_t \sim N(0,1)$$

$$\text{cov}(\varepsilon_s, \varepsilon_t) = 0 \text{ for } s \neq t$$

and $d_{It}(0)$ is the risk-free return in year t.

Equilibrium asset-category model

Initial values required:

$$Y_{I0}(s) = sy_I(s) \text{ and } Y_{C0}(s) = sy_C(s)$$

where $y_I(s)$ and $y_C(s)$ are the yields on index-linked and conventional zero-coupon bonds respectively.

$$\delta_{I1}(0) = Y_{I0}(1)$$

$$\sigma_{IM}(s) = -\sigma_M \left\{ b_{I1}(s) + b_{I2}(s) \right\}$$

$$\sigma_{CM}(s) = -\sigma_M \left\{ b_\gamma + b_{C1}(s) + b_{I2}(s) \right\} \qquad b_\gamma = -0{,}013\ 79$$

$$\sigma_{EM} = b_{E1}\sigma_M \qquad b_{E1} = 0{,}139\ 23$$

$$\mu_{M1} = g\,\delta_{I1}(0) \text{ for } \delta_{I1}(0) > 0;$$

$$\delta_{I1}(0) \text{ otherwise}$$

Equilibrium asset-category model

For $t = 1,\ldots, T$:

$$k_t = \frac{\mu_{Mt} - \delta_{It}(0)}{\sigma_M^2}$$

$$\eta_{jt} = \sum_{i=1}^{N} a_{ij}\varepsilon_{it}$$

$$\mu_{\gamma t} = Y_{C,t-1}(1) - Y_{I,t-1}(1) - \phi$$

$$\mu_{It}(s) = \delta_{It}(0) + k_t\sigma_{IM}(s)$$

$$\mu_{Ct}(s) = \delta_{It}(0) + k_t\sigma_{CM}(s)$$

$$\mu_{Et} = \delta_{It}(0) + k_t\sigma_{EM}$$

Equilibrium asset-category model

For t = 1,…, T ctd.:

$$\gamma_t = \mu_{\gamma t} + b_\gamma \eta_{3t}$$

$$\delta_{It}(s) = \mu_{It}(s) - b_{I1}(s)\eta_{1t} - b_{I2}(s)\eta_{2t}$$

$$\delta_{Ct}(s) = \mu_{Ct}(s) - b_\gamma \eta_{3t} - b_{C1}(s)\eta_{4t} - b_{C2}(s)\eta_{5t}$$

$$\delta_{Et} = \mu_{Et} + b_{E1}\eta_{6t}$$

$$Y_{It}(s) = Y_{I,t-1}(s+1) - \delta_{It}(s)$$

$$Y_{Ct}(s) = Y_{C,t-1}(s+1) - \gamma_t - \delta_{C,t}(s)$$

$$\delta_{I,t+1}(0) = Y_{It}(1)$$

Liabilities specification & model

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Specification:
For a member aged x in service at time t, we let P_{xt} denote the pension accrued for service to time t, conditional on information at that time. We define the accrued pension at time 0:

$$P_{x0} = \pi n S_{x0}$$

where:

$\pi = 0{,}02$ is the rate of pension accrual per year of service;
n is the length of service of that member in years from date of entry to time 0; and
S_{xt} is the member's annual salary during year $t + 1$.

Liabilities specification & model

Specification (ctd.):
When the member retires the accrued pension:

$$P_{x,R-x-1} = \pi n S_{x,R-x-1}$$

becomes payable, where $R = 65$.

If P_{xt} is payable at time t then we define the cash flow in year t as:

$$c_{xt} = \tfrac{1}{2}\left(p_{x,t-1} + p_{xt}\right) P_{xt}$$

where p_{xt} is the probability that the pensioner will be alive at time t.

Liabilities specification & model

Specification (ctd.):

Accrued pensions			Accruing pensions		
age cohort	no. of members	pensions (R'000)	age cohort	no. of members	pensions (R'000)
x	N_x	P_x	x	N_x	P'_x
25	360	2 394	25	360	642
35	1680	28 844	35	1680	3 480
45	2040	51 896	45	2040	4 230
55	1804	54 706	55	1804	3 402
65	1442	64 932	62	762	1 288
75	1010	49 442			
85	600	20 864			

Liabilities specification & model

- no exits before retirement
- mortality only after retirement
- projected unit method
- salaries and pensions expressed in real terms

Liabilities specification & model

Salaries model

$$S_{mt} = S_{m,t-1} \exp\left(\xi_t + \bar{\zeta}_t\right)$$

$$\xi_t = \mu_\xi + b_{\xi 1}\eta_{3t} + b_{\xi 2}\eta_{7t} + \sigma_\xi \varepsilon_{\xi t}$$

$$\bar{\zeta}_t = \mu_{\varsigma x} + \sigma_{\bar{\varsigma} x}\varepsilon_{\varsigma t}$$

$$\mu_\xi = 0,01 \quad b_{\xi 1} = -0,005 \quad b_{\xi 2} = 0,005 \quad \sigma_\xi = 0,03$$

$$\mu_{\varsigma x} = \alpha_{\mu\varsigma} + \beta_{\mu\varsigma}\exp\left(-\lambda_{\mu\varsigma} x\right)$$

$$\alpha_{\mu\varsigma} = 0,016 \quad \beta_{\mu\varsigma} = 0,5 \quad \lambda_{\mu\varsigma} = 0,1$$

$$\sigma^2_{\bar{\zeta} x} = \frac{\sigma^2_{\zeta x}}{M_{x,t-1}}$$

$$\sigma_{\varsigma x} = \alpha_{\sigma\varsigma} + \beta_{\sigma\varsigma}\exp\left(-\lambda_{\sigma\varsigma} x\right)$$

$$\alpha_{\sigma\varsigma} = 0,042 \quad \beta_{\sigma\varsigma} = 0,5 \quad \lambda_{\sigma\varsigma} = 0,08$$

Liabilities specification & model

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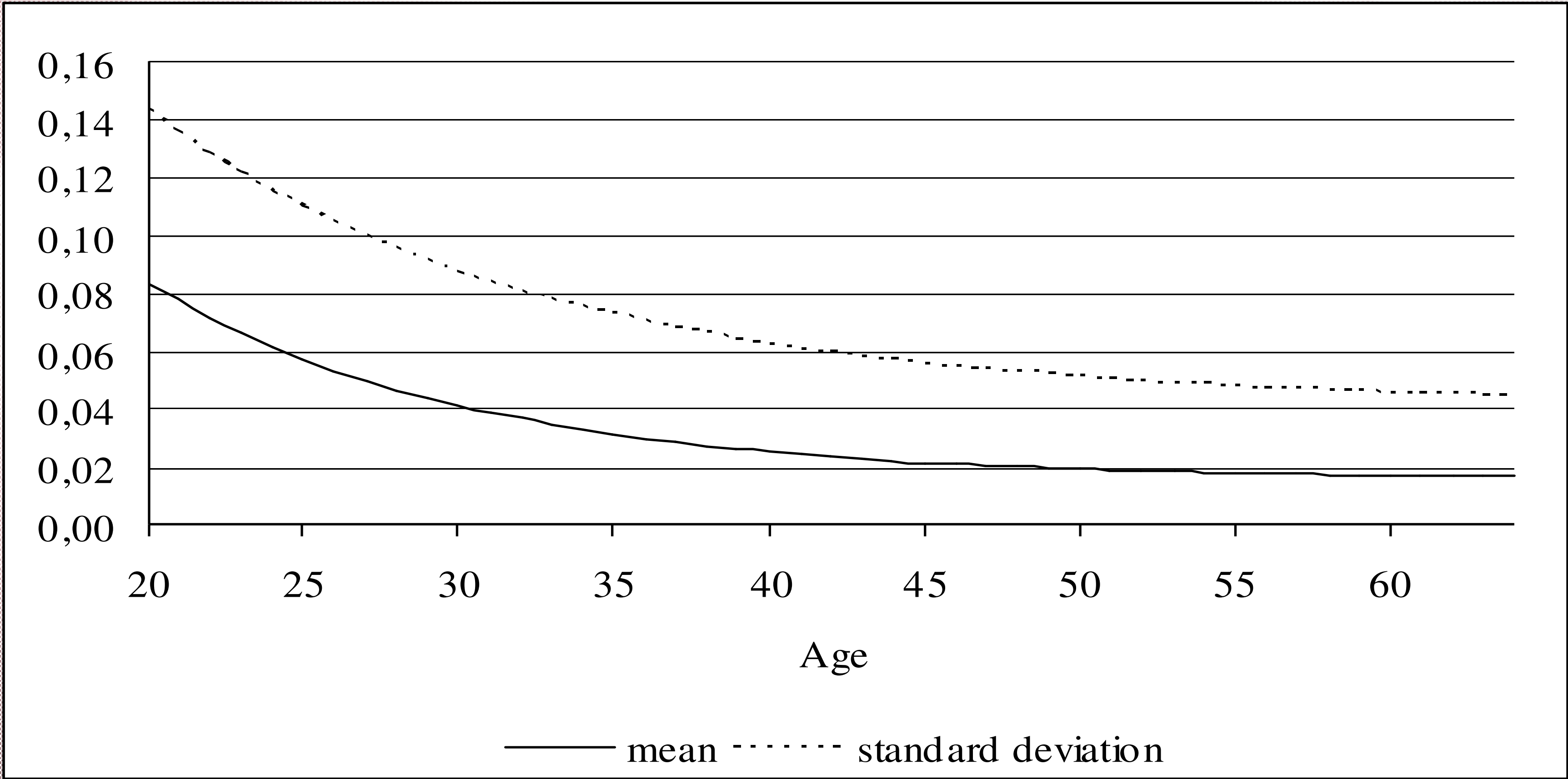
Salaries model: member-specific increases



Liabilities specification & model

Pension increases:

$$P_t = P_{t-1} \exp\left\{\max\left(0, -\gamma_t\right)\right\}$$

Liabilities specification & model

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Pensioner mortality:

$$\nu_{\{x\}}^{SAP98} = \frac{\nu_{\{x\}}^{PNL00}}{\nu_{\{x\}}^{IL00}} \nu_{\{x\}}^{SAIL98}$$

$$\nu_{\{x\}}^{SAP} = \nu_{\{x\}}^{SAP98} \exp\left(10\mu_{\nu}\right)$$

$$\nu_{\{x\}+t}^{SAP} = \nu_{\{x\}+t-1}^{SAP} \exp\left(\chi_{\nu t}\right)$$

where:

$$\chi_{\nu t} = \chi_{\nu,t-1} + \mu_{\nu} + b_{\nu}\eta_{7t} + \sigma_{\nu}\varepsilon_{\nu t}$$

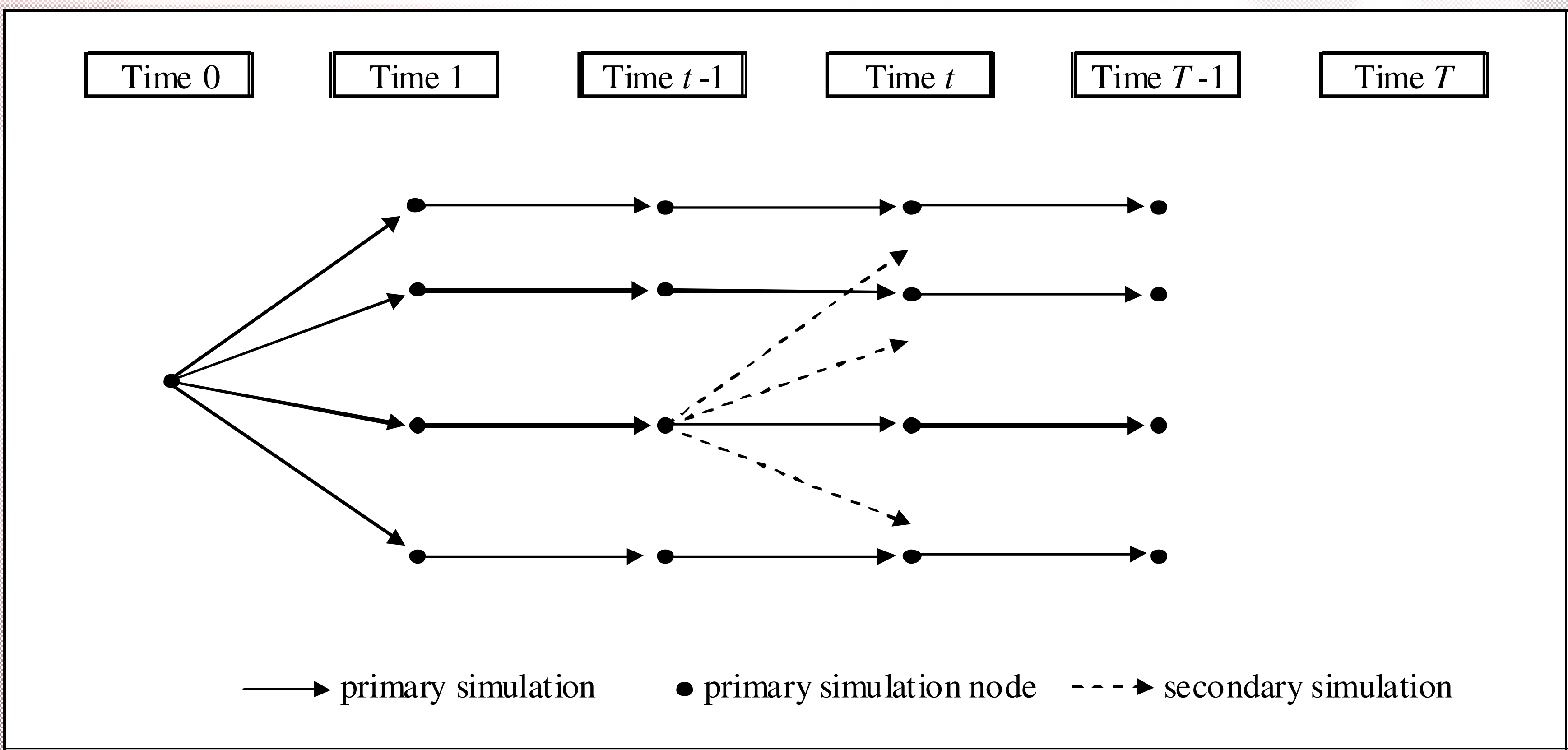
$\mu_{\nu} = -0,004$

$b_{\nu} = -0,001$

$\sigma_{\nu} = 0,005$

Pricing method

Primary and secondary simulations:



Pricing method

State-space vector:

$$\boldsymbol{x}_t = \begin{pmatrix} P_{It}(s_1) \\ \vdots \\ P_{It}(s_u) \\ P_{Ct}(s_1) \\ \vdots \\ P_{Ct}(s_u) \\ \theta_t \\ P_{x_1 t} \\ \vdots \\ P_{x_N t} \end{pmatrix}$$

where:

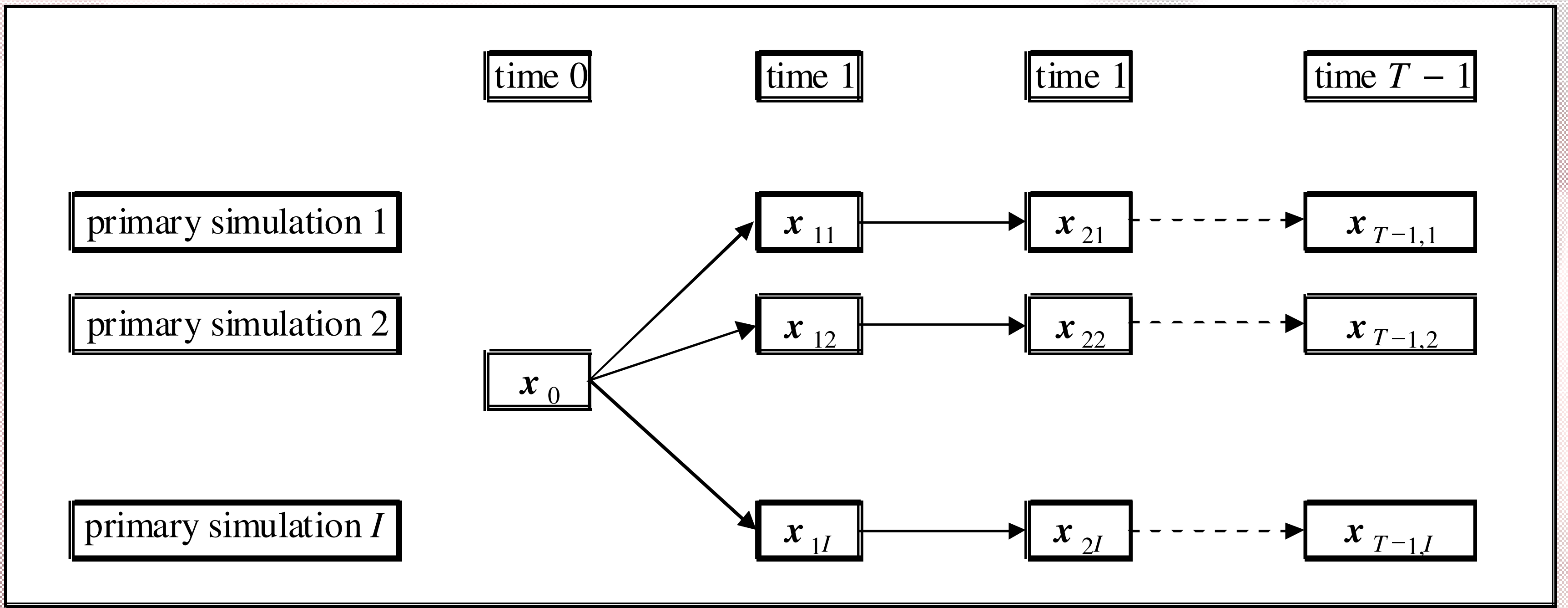
$$P_{It}(s) = \exp\{-Y_{It}(s)\}$$

$$P_{Ct}(s) = \exp\{-Y_{Ct}(s)\}$$

$$\theta_t = \exp(\chi_{vt})$$

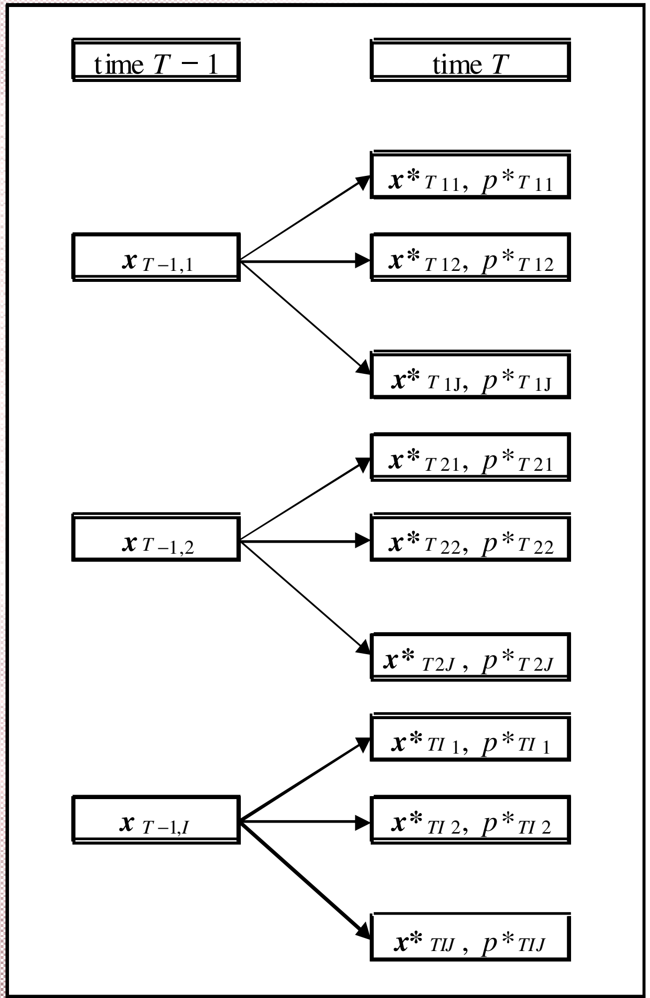
Pricing method

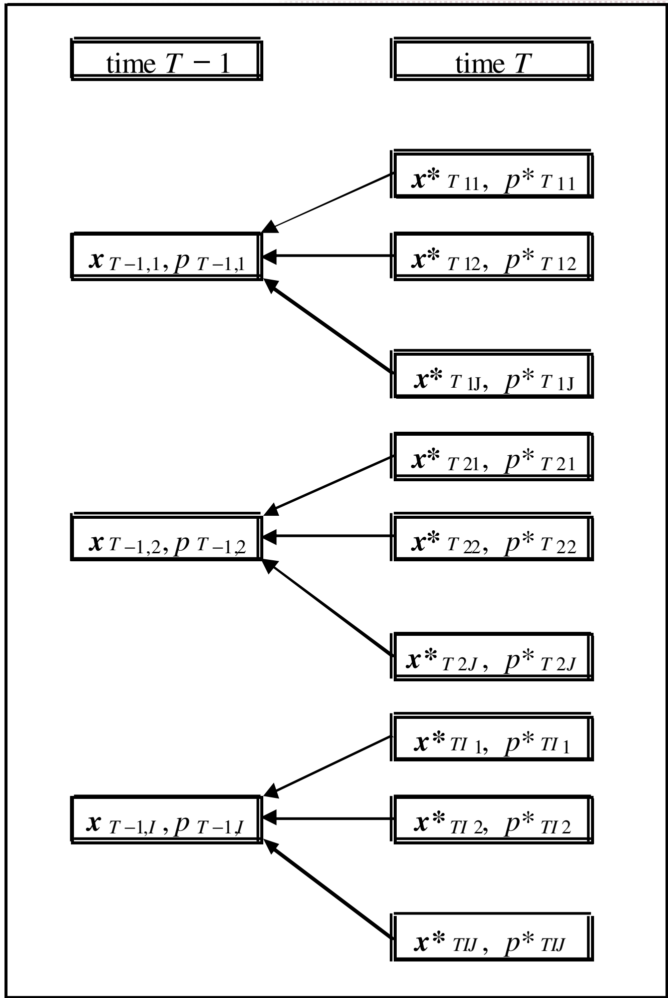
Primary simulations of the state-space vector:



Pricing method

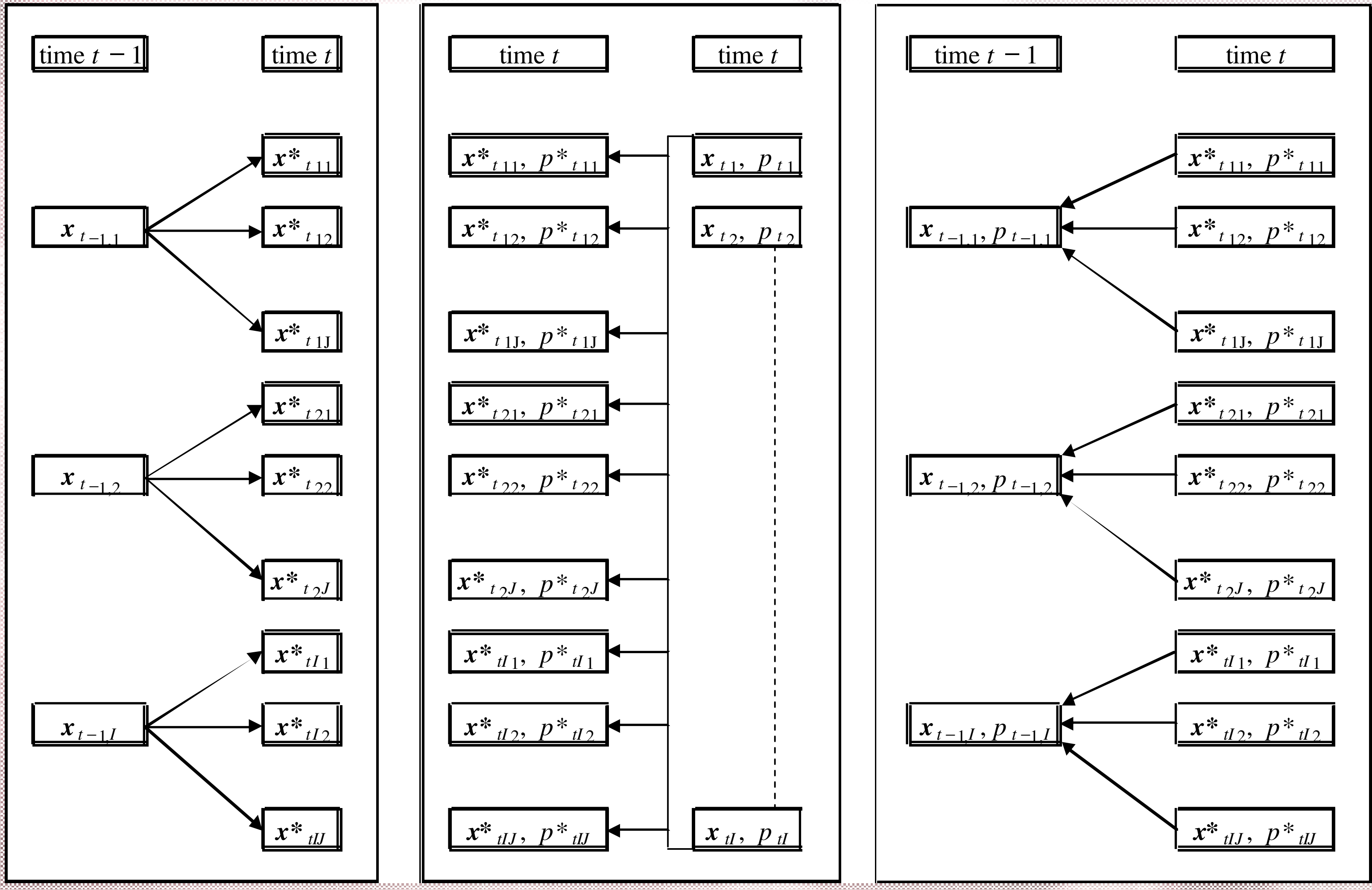
Secondary simulations: final year





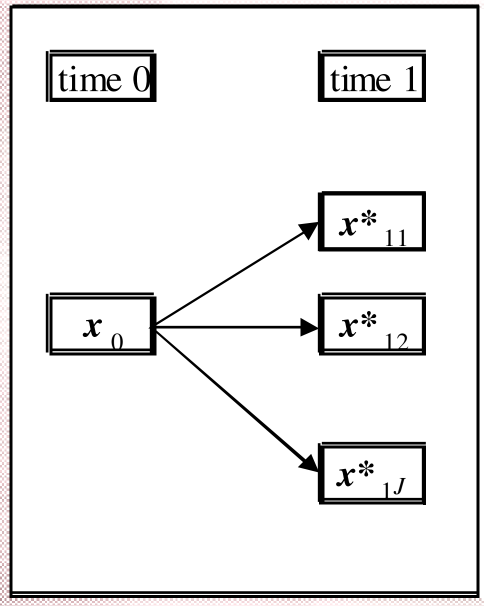
Pricing method

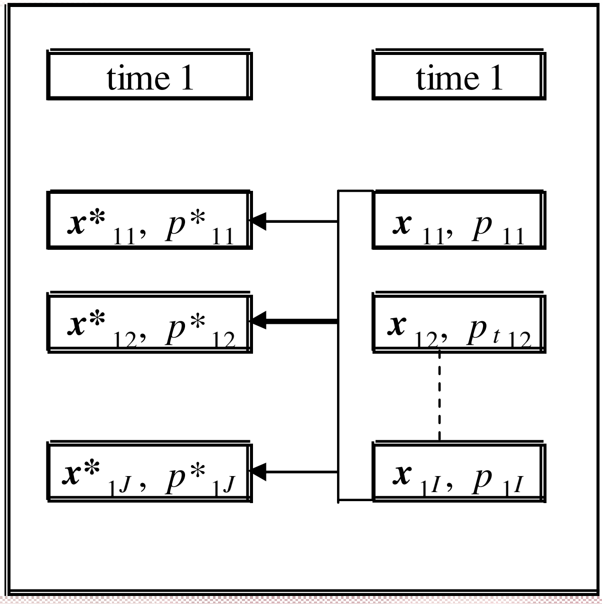
Secondary simulations: year t

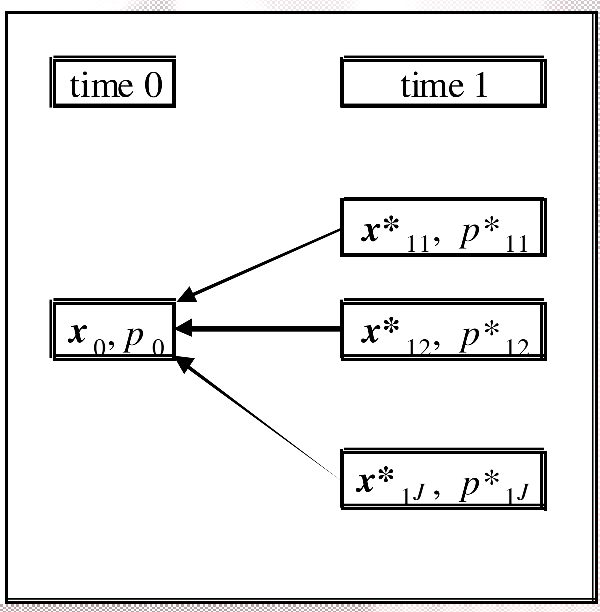


Pricing method

Secondary simulations: year 1







Pricing method

$$\hat{\sigma}_{\varepsilon t}^2 = \hat{\sigma}_{Ft}^2 - \hat{\boldsymbol{\sigma}}'_{FVt}\hat{\boldsymbol{\Sigma}}_{Vt}^{-1}\hat{\boldsymbol{\sigma}}_{FVt}$$

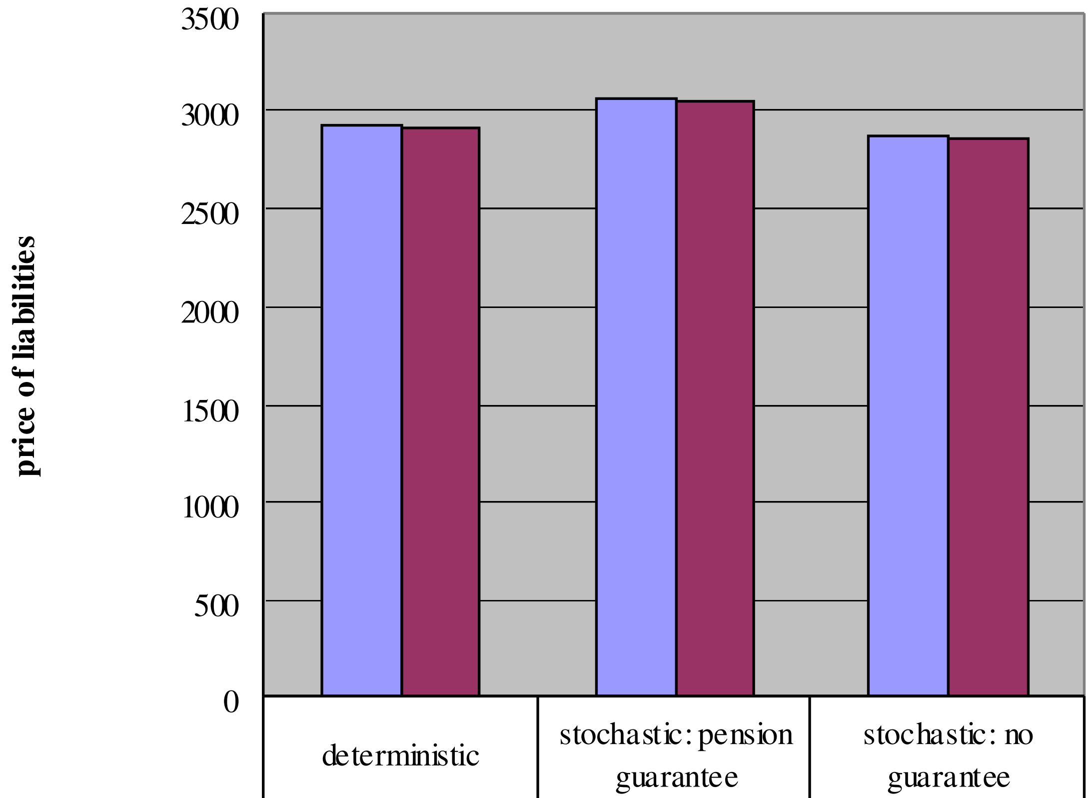
$$z_t = \hat{\boldsymbol{\Sigma}}_{Vt}^{-1}\left(\hat{\boldsymbol{\mu}}_{Vt} - f_t\mathbf{1}\right) \qquad \boldsymbol{m}_t = \frac{1}{z_t'\mathbf{1}} z_t$$

$$\hat{\mu}_{Mt} = \boldsymbol{m}'_t\hat{\boldsymbol{\mu}}_{Vt} \qquad \hat{\sigma}_{Mt}^2 = \boldsymbol{m}'_t\hat{\boldsymbol{\Sigma}}_{Vt}\boldsymbol{m}_t \qquad \hat{\sigma}_{HMt} = \boldsymbol{m}'_t\hat{\boldsymbol{\sigma}}_{FVt}$$

$$\hat{\beta}_{Ft}^* = \frac{\hat{\sigma}_{HMt} + \hat{\sigma}_{\varepsilon t}\hat{\sigma}_{Mt}}{\hat{\sigma}_{Mt}^2}$$

$$P_{L,t-1} = \frac{1}{f_t}\left\{\hat{\mu}_{Ft} - \hat{\beta}_{Ft}^*\left(\hat{\mu}_{Mt} - f_t\right)\right\}$$

Results and sensitivity



	deterministic	stochastic: pension guarantee	stochastic: no guarantee
Model points	2930	3063	2874
Fund data	2912	3044	2856

Pricing method

Results and sensitivity

Sex	Age	Value per unit accrued pension: deterministic valuation	stochastic price: 1 member	1 member % incr	entire cohort	entire cohort % incr	Aggregate value: deterministic valuation R'million	stochastic price R'million	% incr
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Female									
	25	14,11	16,40	16,2	16,25	-0,9	17	19	15,2
	35	11,94	13,79	15,5	13,70	-0,7	172	198	14,7
	45	12,00	13,81	15,1	13,72	-0,6	311	356	14,3
	55	12,67	13,96	10,2	13,88	-0,5	347	380	9,6
	65	13,36	13,41	0,4	13,41	0,0	434	435	0,4
	75	9,53	9,25	-2,9	9,25	0,0	236	229	-2,9
	85	5,96	5,53	-7,2	5,53	0,0	62	58	-7,2
	total						1 579	1 675	6,1
Male									
	25	12,27	14,05	14,5	13,92	-0,9	15	17	13,4
	35	10,35	11,77	13,8	11,69	-0,7	149	169	13,0
	45	10,37	11,75	13,3	11,67	-0,6	269	303	12,6
	55	10,89	11,83	8,7	11,77	-0,5	298	322	8,1
	65	11,38	11,26	-1,1	11,26	0,0	369	366	-1,1
	75	7,95	7,60	-4,3	7,60	0,0	196	188	-4,3
	85	5,25	4,80	-8,5	4,80	0,0	55	50	-8,5
	total						1 351	1 414	4,6
Total							2 930	3 088	5,4
Aggregate							2 930	3 063	4,5
Adjusted to fund data							2 912	3 044	4,5

Results and sensitivity

		Parameter			Test result (R'million)	
set	name	description	standard value	test value	price (R'million)	change in price
0		standard values			3 063	N/A
1	$b_{\xi 1}$	general salary increase: sensitivity to inflation	–0,005	0	3 067	0,14
2	$b_{\xi 2}$	general salary increase: sensitivity to return on market portfolio	0,005	0	3 063	-0,01
3	σ_{ξ}	general salary increase: residual volatility	0,03	0,01	3 063	0,01
4	$\alpha_{\sigma\varsigma}$	volatility of additional increase: level parameter	0,042	0	3 064	0,01
	$\beta_{\sigma\varsigma}$	volatility of additional increase: slope parameter	0,5	0		
	$\lambda_{\sigma\varsigma}$	volatility of additional increase: age parameter	0,08	0		
5	b_{ν}	pensioner mortality: sensitivity to return on market portfolio	–0,001	0	3 063	0,01
	σ_{ν}	pensioner mortality: residual volatility	0,005	0,001		
6	g	return on market portfolio: sensitivity to risk-free rate	1,39	1,2	3 052	-0,37
7	σ_M	return on market portfolio: residual volatility	0,159	0,1	3 077	0,45
8	b_{γ}	force of inflation: residual volatility	–0,01379	0	3 030	-1,07
9	ϕ	inflation risk premium	0,003	0	3 063	0,00
10	b_{E1}	return on equities: residual volatility	0,13923	0,1	3 063	0,00

Conclusions

- Method computationally demanding, but not impossible: 47 hours.
- Convergence complicated, but Sobol numbers expedite it.
- Stochastic price 4,5% higher than deterministic: because of pension guarantee (may be overstated).
- Without guarantee, stochastic price only 1,9% less than deterministic: *If the valuation of the liabilities should allow for a risk premium only to the extent that the trustees are unable to avoid risk, then the valuation basis must be much closer to a risk-free basis than that produced by the risk premiums typically used.*
- Effects of non-additivity: intra-cohort 0–0,9% plus inter-cohort 0,8%.
- Major sensitivities:
  - volatility of force of inflation in excess of conditional ex-ante expected inflation
  - sensitivity of ex-ante expected returns on the market portfolio to positive risk-free returns
  - residual volatility of the return on the market portfolio.
- Overall effect: *Excluding uncertainties common to deterministic and stochastic valuations, an error of about 4,5% is reduced to uncertainty of about 1%.*

Contact details

Rob Thomson

rthomson@icon.co.za

011 646 5332

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2009 Convention Lite and the Pensions, Health and Life Seminars

19-20 May 2009

Sandton Convention Centre Johannesburg, South Africa

Pricing method

$$p_j = \frac{\tilde{\boldsymbol{w}}_j{}' \tilde{\boldsymbol{p}}_j}{\tilde{\boldsymbol{w}}_j{}' \mathbf{1}}$$

where:

$$w_{ij} = \frac{1}{\sum_{d=1}^{D} \frac{r_d}{s_d^{(n)}} \left| x_{dj} - x_{di}^* \right|^n}$$

$$s_d^{(n)} = \frac{1}{I-1} \sum_{i=1}^{I} \left| x_{di}^* - \bar{x}_d^* \right|^n$$

$$\begin{pmatrix} r_1 \\ \vdots \\ r_D \end{pmatrix} = \mathbf{R}^{-1}\mathbf{1}$$

$$\mathbf{R} = \begin{pmatrix} 1,001 & \hat{\rho}_{12} & \cdots & \hat{\rho}_{1D} \\ \hat{\rho}_{21} & 1,001 & \cdots & \hat{\rho}_{2D} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\rho}_{D1} & \hat{\rho}_{D2} & \cdots & 1,001 \end{pmatrix}$$

Results and sensitivity

Convergence: